MECA-H-406 Composite structures - Exercises 4: solutions

Exercise 1

The Tsaï-Hill criterion for the maximum work theory states that if eq.(1) is verified, the ply resists the corresponding load :

$$\left(\frac{\sigma_L}{\sigma_{LU}}\right)^2 + \left(\frac{\sigma_T}{\sigma_{TU}}\right)^2 - \frac{\sigma_L}{\sigma_{LU}} \cdot \frac{\sigma_T}{\sigma_{LU}} + \left(\frac{\tau_{LT}}{\tau_{LTU}}\right)^2 \leq 1 \quad . \tag{1}$$

Note : the terms of this formula must be adapted depending on the sign of the normal stress in the L and T directions (traction or compression). The signs are adapted so that each ratio is positive (we are dealing with energies) ; consequently, in the ratio σ_T/σ_{LU} of the third term, the choice of the value of σ_{LU} is based on the state (traction or compression) in the T direction.

Eq.(1) must be evaluated in the $\{L, T\}$ frame so in each case we start by expressing the stress in these axes, using the transformation matrix (see exercise sessions 2 and 3). In what follows, we denote the left-hand term of eq.(1) by s. The results are summarized below.

(a) $\sigma_{LT} = \left[\begin{array}{ccc} 210 & -210 & 0 \end{array}\right]^T$ [MPa] $\Rightarrow$ $s = 0.83$ $\Rightarrow$ the ply resists

(b) $\sigma_{LT} = \left[\begin{array}{ccc} -210 & 210 & 0 \end{array}\right]^T$ [MPa] $\Rightarrow$ $s = 17.64$ $\Rightarrow$ the ply breaks

(c) $\sigma_{LT} = \left[\begin{array}{ccc} 15.7 & -24.4 & -2.5 \end{array}\right]^T$ [MPa] $\Rightarrow$ $s = 0.24$ $\Rightarrow$ the ply resists

Note : the superscript T denotes the transpose.

Exercise 2

For an orthotropic material, the thermal strains are defined in the $\{L, T\}$ frame as

$$\begin{pmatrix} \epsilon_L^t \\ \epsilon_T^t \\ \frac{1}{2}\,\gamma_{LT}^t \end{pmatrix} = \begin{pmatrix} \alpha_L \\ \alpha_T \\ 0 \end{pmatrix} \Delta t \quad , \tag{2}$$

where the superscript t refers to thermal effects and Δt is the (relative) temperature change with respect to the reference temperature of the laminate. One can also write the equation governing the transformation from the $\{x, y\}$ to the $\{L, T\}$ frame :

$$\begin{pmatrix} \epsilon_L^t \\ \epsilon_T^t \\ \frac{1}{2}\,\gamma_{LT}^t \end{pmatrix} = [T] \,.\, \begin{pmatrix} \epsilon_x^t \\ \epsilon_y^t \\ \frac{1}{2}\,\gamma_{xy}^t \end{pmatrix} \quad . \tag{3}$$

Combining eq.(2) and (3) and using the properties of the transformation matrix $[T]$, one can write

$$\begin{pmatrix} \epsilon_x^t \\ \epsilon_y^t \\ \frac{1}{2}\,\gamma_{xy}^t \end{pmatrix} = [T\,(\theta)]^{-1} \,.\, \begin{pmatrix} \alpha_L \\ \alpha_T \\ 0 \end{pmatrix} \Delta t = [T\,(-\theta)] \,.\, \begin{pmatrix} \alpha_L \\ \alpha_T \\ 0 \end{pmatrix} \Delta t = \begin{pmatrix} \alpha_x \\ \alpha_y \\ \frac{1}{2}\,\alpha_{xy} \end{pmatrix} \Delta t \quad . \tag{4}$$

For the numerical values considered, one obtains

$$\alpha_x = 4.67e-6 \quad ; \quad \alpha_y = 22.9e-6 \quad ; \quad \alpha_{xy} = 21.8e-6 \quad [1/K] \quad . \tag{5}$$

Exercise 3

From the numerical data, one can easily write the compliance matrix in the $\{L,T\}$ frame, $[S]$ (see exercise sessions 2 and 3) :

$$[S] = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & 0 \\ -\nu_{LT}/E_L & 1/E_T & 0 \\ 0 & 0 & 1/G_{LT} \end{bmatrix} \quad [1/Pa] \quad , \tag{6}$$

from which one computes the corresponding stiffness matrix in the $\{L,T\}$ frame, $[Q]$:

$$[Q] = [S]^{-1} \quad [Pa] \quad . \tag{7}$$

It is then possible to write $\left[\overline{Q}\right]$, the stiffness matrix in the $\{x,y\}$ frame, from the corresponding equations (slide 16 of part 3 or eq.(5.61) of the reference book). For this example, one obtains

$$\left[\overline{Q}\right]_{+30^o} = \begin{bmatrix} 27.9 & 7.7 & 10.4 \\ 7.7 & 11.1 & 4.1 \\ 10.4 & 4.1 & 9.5 \end{bmatrix} \quad ; \quad \left[\overline{Q}\right]_{-30^o} = \begin{bmatrix} 27.9 & 7.7 & -10.4 \\ 7.7 & 11.1 & -4.1 \\ -10.4 & -4.1 & 9.5 \end{bmatrix} \quad , \quad [\text{GPa}]$$

$$\left[\overline{Q}\right]_{+15^o} = \begin{bmatrix} 37.6 & 4.1 & 7.3 \\ 4.1 & 8.6 & 1.1 \\ 7.3 & 1.1 & 5.9 \end{bmatrix} \quad ; \quad \left[\overline{Q}\right]_{-15^o} = \begin{bmatrix} 37.6 & 4.1 & -7.3 \\ 4.1 & 8.6 & -1.1 \\ -7.3 & -1.1 & 5.9 \end{bmatrix} \quad , \quad [\text{GPa}]$$

The stiffness matrices A, B and D are computed from their definition (part 5 of the slides or chapter 6.3 of the reference book or exercise session 3)

$$A = \sum_{i=1}^{k} \left[\overline{Q}\right]_k \cdot \left(h_k - h_{k-1}\right) = \begin{bmatrix} 168.5 & 27.6 & 0 \\ 27.6 & 47.9 & 0 \\ 0 & 0 & 36.7 \end{bmatrix} .1e6 \quad , \tag{8}$$

$$B = \frac{1}{2} \sum_{i=1}^{k} \left[\overline{Q}\right]_k \cdot \left(h_k^2 - h_{k-1}^2\right) = \begin{bmatrix} 14.6 & -5.4 & 1.5 \\ -5.4 & -3.8 & -6.3 \\ 1.5 & -6.3 & -5.4 \end{bmatrix} .1e3 \quad , \tag{9}$$

$$D = \frac{1}{3} \sum_{i=1}^{k} \left[\overline{Q}\right]_k \cdot \left(h_k^3 - h_{k-1}^3\right) = \begin{bmatrix} 348.6 & 58.4 & 66.3 \\ 58.4 & 100.4 & 19.5 \\ 66.3 & 19.5 & 77.4 \end{bmatrix} \quad . \tag{10}$$

Exercise 4

Particular choices for the construction of a laminate can lead to simplifications in the equations that relate the loads applied to the laminate to the corresponding strains :

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \cdot \begin{pmatrix} \epsilon^0 \\ k \end{pmatrix} \quad , \tag{11}$$

where N and M are respectively the in-plane (membrane) and the out-of-plane (bending) loads applied to the laminate and ϵ^0 and k are respectively the corresponding membrane strains and curvatures. These simplifications correspond mostly to cancellation of terms of the A, B and D matrices, or to particular relations between some of their terms. They are described in slides 14-18 of part 5 and in chapter 6.4 of the reference book. By examining the laminates, we can draw the following conclusions :

(a)
- equilibrated angle-ply laminate ($A_{16} = A_{26} = 0$) $\Rightarrow$ in-plane behavior is orthotropic (membrane longitudinal and shear strains are decoupled)
- non-symmetrical ($B \neq \mathbb{O}_{3\times3}$) $\Rightarrow$ in-plane and out-of-plane effects are coupled
- non-anti-symmetrical (D_{16} and $D_{26} \neq 0$) $\Rightarrow$ bending and torsion are coupled
- non-quasi-isotropic $\Rightarrow$ the in-plane response is a function of the orientation of the laminate with respect to the load

(b)
- equilibrated angle-ply laminate ($A_{16} = A_{26} = 0$)
- symmetrical ($B = \mathbb{O}_{3\times3}$) $\Rightarrow$ in-plane and out-of-plane effects are decoupled
- non-anti-symmetrical (D_{16} and $D_{26} \neq 0$)
- non-quasi-isotropic

(c)
- equilibrated angle-ply laminate ($A_{16} = A_{26} = 0$)
- non-symmetrical ($B \neq \mathbb{O}_{3\times3}$)
- anti-symmetrical ($D_{16} = D_{26} \neq 0$) $\Rightarrow$ bending and torsion are decoupled
- non-quasi-isotropic

The response to unidirectional loads can thus be summarized as follows.

		$N_x > 0$	$M_x > 0$
(a)	ϵ_x^0	> 0	$\neq 0$
	ϵ_y^0	$\neq 0$	$\neq 0$
	γ_{xy}^0	0	$\neq 0$
	k_x	$\neq 0$	> 0
	k_y	$\neq 0$	$\neq 0$
	k_{xy}	$\neq 0$	$\neq 0$
(b)	ϵ_x^0	> 0	0
	ϵ_y^0	$\left(-\nu_{xy}\epsilon_x^0\right)^*$	0
	γ_{xy}^0	0	0
	k_x	0	> 0
	k_y	0	$\neq 0$
	k_{xy}	0	$\neq 0$
(c)	ϵ_x^0	> 0	$\neq 0$
	ϵ_y^0	$\neq 0$	$\neq 0$
	γ_{xy}^0	0	$\neq 0$
	k_x	$\neq 0$	> 0
	k_y	$\neq 0$	$\neq 0$
	k_{xy}	$\neq 0$	0

* Poisson's effect : the sign and the amplitude depend on those of the apparent Poisson's ratio ν_{xy}.

MECA-H-406 Composite Strucures
Exercises 4: Solutions, additional material

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1 Exercise 3 - detailing the computations

The first important point here is to understand $[Q] \neq [\bar{Q}]$. The expression of $[\bar{Q}]$ can be obtained in a systematic manner from the expression of $[Q]$ by introducing two additional matrices due to the 1/2 factor that might appear on the strain vectors,

$$[\bar{Q}(\theta)] = \underbrace{\underbrace{[T(\theta)]^{-1}.[Q].\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{vmatrix}.[T(\theta)]}_{\text{using part \# 3, slide \# 15}}.\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{vmatrix}}_{\text{using part \# 3, slide \# 16}} \quad (1)$$

The definition of matrix **A** is,

$$A_{ij} = \sum_{k-1}^{n} (\bar{Q}_{ij})_k (h_k - h_{k-1}) \quad (2)$$

The next important point is the definition of the h_k with respect to the vertical axis z, see figures on part # 5 slides # 9 and # 12. In application to the exercise:

$$h_0 = -2.5mm$$

$$h_1 = -1.5mm$$

$$h_2 = 0mm$$

$$h_3 = 1mm$$

$$h_4 = 2.5mm$$

The expanded expression of matrix **A** is then,

$$A = [\bar{Q}(\theta)]_{+30\text{deg}}.(h_1 - h_0) + [\bar{Q}(\theta)]_{-15\text{deg}}.(h_2 - h_1) + [\bar{Q}(\theta)]_{-30\text{deg}}.(h_3 - h_2) + [\bar{Q}(\theta)]_{+15\text{deg}}.(h_4 - h_3) \quad (3)$$

Numerically, one should find:

$$A = \begin{vmatrix}168.5 & 27.6 & 0\\ 27.6 & 47.9 & 0\\ 0 & 0 & 36.7\end{vmatrix}.10^6 \quad (4)$$

Similarly one can obtain **B** and **D** matrices.