MECA-H-406 Composite structures - Exercises 3: solutions

Exercise 1

(a)

• Approach 1

The compliance matrix, $\left[\,\overline{S}\,\right]$ relates the strains and the stress in the structural axes $\{x, y\}$ according to eq.(1) where the particular load case has been taken into account (with $\sigma_x = \sigma$) :

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = \left[\,\overline{S}\,\right] . \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \overline{S}_{11} \\ \overline{S}_{12} \\ \overline{S}_{66} \end{pmatrix} . \sigma \tag{1}$$

The components of $\left[\,\overline{S}\,\right]$ are given in eq.(5.63) of the reference book or in the course slides (slide 17, part 3) and must be injected in eq.(1) to serve as a basis for the reasoning.

• Approach 2

We focus on the behavior as seen from the $\{L, T\}$ frame, particularized for an orientation of 45^o :

$$\begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = [T] . \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \cos\theta^2 \\ \sin\theta^2 \\ -\sin\theta\cos\theta \end{pmatrix} . \sigma = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} . \frac{\sigma}{2} \tag{2}$$

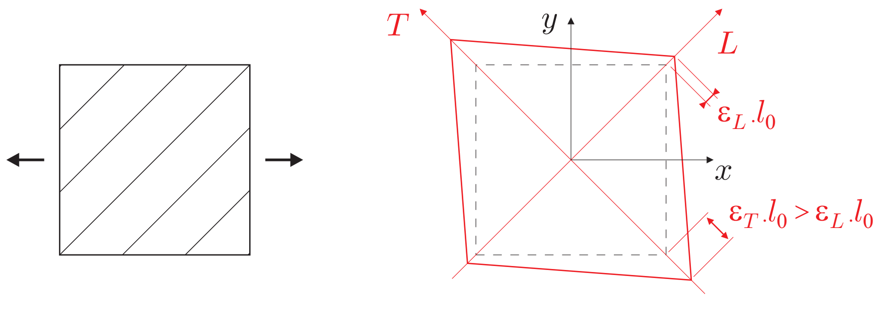
In the framework of the course, we focus on unidirectional orthotropic laminates for which the following relations apply

$$E_L \geq E_T \quad \text{and} \quad \nu_{LT} \geq \nu_{TL} \quad , \tag{3}$$

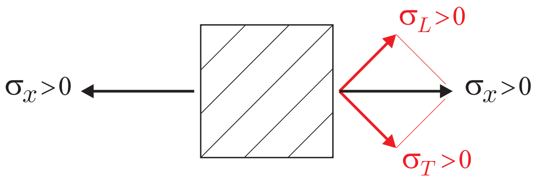
which leads to

$$\epsilon_T \geq \epsilon_L > 0 \quad \text{and} \quad \gamma_{LT} < 0 \quad , \tag{4}$$

which corresponds to deformation (3), as illustrated below.



• Approach 3

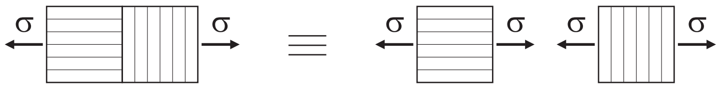


As illustrated above, the load can be decomposed in two equal contributions of the same sign and magnitude in the $\{L, T\}$ frame. However, the stiffness seen by each load component is very different :

- in the L direction, the fibers and the matrix act like springs in parallel ; hence, the fibers act as stiffeners.
- in the T direction, the fibers and the matrix act like springs in series ; hence as the matrix is much softer, it tends to be the limiting element in the transverse stiffness.

In the framework of the course, we focus on unidirectional orthotropic laminates for which $E_f \geq E_m$, leading to $\epsilon_L \geq \epsilon_T > 0$ for a positive load. The resulting diamond-like shape corresponds to $\gamma_{LT} < 0$ with respect to the sign conventions for the shear terms (slide 16 of part 4 or Fig.4.17 of the book).

(b)



The static equilibrium is illustrated above : each ply undergoes the same load, only the orientation relative to the direction of the fibers changes. For what follows, we can either apply the formulas to work in the $\{x, y\}$ frame, or work in the $\{L, T\}$ frame and use the simple identity rules relating the strains in both frames for the particular orientations of these plies. The second approach is applied here.

• Ply 1

By particularizing the definition of the transformation matrix $[T]$ for $\theta = 0$ and combining it with the stress-strain relationship in the $\{L, T\}$ frame, one obtains :

$$\begin{pmatrix} \epsilon_x^1 \\ \epsilon_y^1 \\ \gamma_{xy}^1 \end{pmatrix} \underset{\theta=0}{=} \begin{pmatrix} \epsilon_L^1 \\ \epsilon_T^1 \\ \gamma_{LT}^1 \end{pmatrix} = [S] \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} = \frac{\sigma}{E_L} \begin{pmatrix} 1 \\ -\nu_{LT} \\ 0 \end{pmatrix} \tag{5}$$

• Ply 2

Similary, for $\theta = \pi/2$, one obtains :

$$\begin{pmatrix} \epsilon_y^2 \\ \epsilon_x^2 \\ -\gamma_{xy}^2 \end{pmatrix} \underset{\theta=\pi/2}{=} \begin{pmatrix} \epsilon_L^2 \\ \epsilon_T^2 \\ \gamma_{LT}^2 \end{pmatrix} = [S] \begin{pmatrix} 0 \\ \sigma \\ 0 \end{pmatrix} = \frac{\sigma}{E_T} \begin{pmatrix} -\nu_{TL} \\ 1 \\ 0 \end{pmatrix} \tag{6}$$

Hence, we have $\epsilon_y^1 = \epsilon_y^2 = -\sigma \frac{\nu_{LT}}{E_L}$ which corresponds to case (2).

Exercise 2

Because of the orthotropic behavior, a single load case would not allow to characterize fully the material of the ply. Furthermore, as illustrated below, a load case such as (b) where the natural and structural frames are not aligned is mandatory to account for shear effects (measuring strains for orientations of 0^o and 90^o would keep it hidden, by definition of an orthotropic material).

(a) By particularizing the definition of the transformation matrix $[T]$ for $\theta = 0$ and combining it with the stress-strain relationship in the $\{L, T\}$ frame, one obtains :

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \underset{\theta=0}{=} \begin{pmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{pmatrix} = [S] \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} = \frac{\sigma}{E_L} \begin{pmatrix} 1 \\ -\nu_{LT} \\ 0 \end{pmatrix} \tag{7}$$

which can be combined with the strains measured and one obtains

$$E_L = \frac{\sigma}{\epsilon_x} = 629.4 \text{ [GPa]} \quad \text{and} \quad \nu_{LT} = -\frac{E_L.\epsilon_y}{\sigma} = 0.252 \text{ [/]} \quad . \tag{8}$$

(b) In this step, we express the strains in the $\{L, T\}$ frame both from the transformation of the strains in the $\{x, y\}$ frame, and from the stress-strain relation in the $\{L, T\}$ frame. First, we apply the transformation strains particularized for $\theta = -20^o$:

$$\begin{pmatrix} \epsilon_L \\ \epsilon_T \\ \frac{1}{2}\gamma_{LT} \end{pmatrix} = [T] . \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{pmatrix} = \begin{pmatrix} 0.883.\epsilon_x + 0.117.\epsilon_y - 0.321.\gamma_{xy} \\ 0.117.\epsilon_x + 0.883.\epsilon_y + 0.321.\gamma_{xy} \\ 0.321.\epsilon_x - 0.321.\epsilon_y + 0.383.\gamma_{xy} \end{pmatrix} \quad , \tag{9}$$

where γ_{xy} is an unknown as it hasn't been measured. Then, we write the stress-strain relationship after transforming the stress in the $\{L, T\}$ frame :

$$\begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = [T] . \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.883 \\ 0.117 \\ 0.321 \end{pmatrix} . \sigma \tag{10}$$

leading to

$$\begin{pmatrix} \epsilon_L \\ \epsilon_T \\ \gamma_{LT} \end{pmatrix} = [S] . \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = \begin{pmatrix} \frac{\sigma}{E_L}\left[0.883 - 0.117\nu_{LT}\right] \\ \sigma\left[-0.883\frac{\nu_{LT}}{E_L} + 0.117\frac{1}{E_T}\right] \\ 0.321\frac{\sigma}{G_{LT}} \end{pmatrix} \quad . \tag{11}$$

We can then combine eq.(8), (9) and (11) ; lines 1 to 3 give respectively :

$$\gamma_{xy} = 1344e - 6 \text{ [/]} \quad , \quad E_T = 32.9 \text{ [GPa]} \quad \text{and} \quad G_{LT} = 17.9 \text{ [GPa]} \quad . \tag{12}$$

And using the results above with the equation that relates the major and minor Poisson's ratios, one obtains

$$\nu_{LT} = \nu_{LT}\frac{E_T}{E_L} = 0.013 \text{ [/]} \quad . \tag{13}$$

Exercise 3

(a) In a way similar to Hooke's law, the apparent Young's moduli can be inferred from the equation relating the loads applied to the laminate to the corresponding strains :

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \cdot \begin{pmatrix} \epsilon^0 \\ k \end{pmatrix} \quad , \tag{14}$$

where N and M are respectively the in-plane (membrane) and the bending loads applied to the laminate and ϵ^0 and k are respectively the corresponding membrane strains and curvatures. The stiffness matrices A, B and D are computed from their definition (part 5 of the slides or chapter 6 of the reference book)

$$A = \sum_{i=1}^{k} \left[\overline{Q}\right]_k \cdot (h_k - h_{k-1}) = \begin{bmatrix} 125.3 & 10.94 & 0 \\ 10.94 & 125.3 & 0 \\ 0 & 0 & 13.1 \end{bmatrix} .1e6 \quad , \tag{15}$$

$$B = \frac{1}{2} \sum_{i=1}^{k} \left[\overline{Q}\right]_k \cdot (h_k^2 - h_{k-1}^2) = \mathbb{O}_{3\times 3} \quad , \tag{16}$$

$$D = \frac{1}{3} \sum_{i=1}^{k} \left[\overline{Q}\right]_k \cdot (h_k^3 - h_{k-1}^3) = \begin{bmatrix} 35.65 & 1.89 & 0 \\ 1.89 & 7.65 & 0 \\ 0 & 0 & 2.26 \end{bmatrix} \quad . \tag{17}$$

These numerical results confirm what could be deduced from the inspection of the stacking : the resulting laminate is symmetrical (B=$\mathbb{O}_{3\times 3}$) hence there is no coupling between in-plane and out-of-plane deformations, the laminate is globally orthotropic ($A_{16} = A_{26} = 0$, i.e. normal and shear strains are decoupled) and it decouples bending and torsional effects ($D_{16} = D_{26} = 0$). This makes the determination of the Young's moduli much easier as the in-plane and out-of-plane effects are decoupled ; hence we restrict ourselves to a load case where only membrane forces N are non-zero, reducing eq.(19) to :

$$N = A.\epsilon^0 \quad . \tag{18}$$

By particularizing the definition of N_i to this case where ϵ, and hence σ, do not vary with z, we have

$$N_i = \int_{-h/2}^{+h/2} \sigma_i dz = h.\sigma_i \quad . \tag{19}$$

leading to

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{1}{h} [A] \cdot \begin{pmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} \quad , \tag{20}$$

where the matrix $h^{-1}.[A]$ acts as the stiffness matrix of an orthotropic material ($A_{16} = A_{26} = 0$).

To find E_x, we restrict ourselves to a uniaxial load case :

$$\begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix} = \frac{1}{h} [A] . \begin{pmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} \quad , \tag{21}$$

hence

$$\sigma = \frac{1}{h} \left(A_{11} \epsilon_x^0 + A_{12} \epsilon_y^0 \right) \quad , \tag{22}$$

$$0 = \frac{1}{h} \left(A_{12} \epsilon_x^0 + A_{22} \epsilon_y^0 \right) \quad , \tag{23}$$

$$0 = \gamma_{xy}^0 \quad . \tag{24}$$

After simple manipulations, we obtain the relation between the stress and the strain along the loading direction, x, leading to the apparent Young's modulus E_x :

$$E_x = \frac{\sigma_x}{\epsilon_x^0} = \frac{1}{h} \left(A_{11} - \frac{A_{12}^2}{A_{22}} \right) = 86.34 \quad [GPa] \quad . \tag{25}$$

E_y is determined using the same principles :

$$E_y = \frac{\sigma_x}{\epsilon_x^0} = \frac{1}{h} \left(A_{22} - \frac{A_{12}^2}{A_{11}} \right) = 86.34 \quad [GPa] \quad . \tag{26}$$

Hence we see that the membrane behavior of the laminate is quasi-isotropic.

(b) By following the same procedure, one finds

$$E_x = E_y = 25.12 \quad [GPa] \quad . \tag{27}$$

Again, the membrane behavior of the laminate is quasi-isotropic.

It should be noted that the apparent stiffness in case (b) is much smaller than that of case (a) although the laminates have similar properties. Most of the difference comes from their orientation with respect to the loading direction.