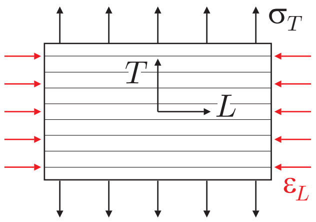
MECA-H-406 Composite structures - Exercises 2: solutions

Exercise 1



(a) The load case is represented in the diagram. The most general approach is based on the compliance matrix, $[S]$, in the natural (orthotropy) axes $\{L, T\}$:

$$\begin{pmatrix} \epsilon_L \\ \epsilon_L \\ \gamma_{LT} \end{pmatrix} = \begin{bmatrix} 1/E_L & -\nu_{LT}/E_L & 0 \\ -\nu_{LT}/E_L & 1/E_T & 0 \\ 0 & 0 & 1/G_{LT} \end{bmatrix} . \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = [S] . \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} \qquad (1)$$

For the load case considered, we have σ_L=0, τ_{LT}=0, σ_T=90MPa, ϵ_L=-2.1e-4 (and $\epsilon_T \neq$0). We can inject these values in eq.(1) ; the first equation gives

$$\epsilon_L = -\frac{\nu_{LT}}{E_L}.\sigma_T \quad \Leftrightarrow \quad \nu_{LT} = -\epsilon_L.\frac{E_L}{\sigma_T} = 0.117 \quad [/] \quad , \qquad (2)$$

where we have used the rule of mixtures for computing $E_L = 50.05$GPa. Note : ϵ_L is negative as the ply is compressed in the longitudinal direction.

Other approach : we first focus on the stress-strain relation in the transverse direction :

$$\epsilon_T = \frac{\sigma_T}{E_T} = 0.0161 \quad [/] \quad , \qquad (3)$$

where we have used Halpin-Tsaï model to approximate E_T=5.59GPa (see session 1). Then, we combine this result with the definition of the minor Poisson's ratio (we have to use this one because the load is in the transverse direction)

$$\epsilon_L = -\nu_{TL}.\epsilon_T \quad \Leftrightarrow \quad \nu_{TL} = -\frac{\epsilon_L}{\epsilon_T} = +1.3e-2 \quad [/] \quad . \qquad (4)$$

Finally, we use the equation that relates the major and the minor Poisson's ratios :

$$\frac{\nu_{LT}}{E_L} = \frac{\nu_{TL}}{E_T} \quad \Leftrightarrow \quad \nu_{LT} = \nu_{TL}\frac{E_L}{E_T} = 0.117 \quad [/] \quad . \qquad (5)$$

Explain why this second approach is less general than the first one.

(b) When a ply is subjected to compression in the longitudinal direction, in most practical cases, the failure comes from the matrix. The composite breaking strain is then given by (slide 11 of part 2, eq.(3.43) of the reference book)

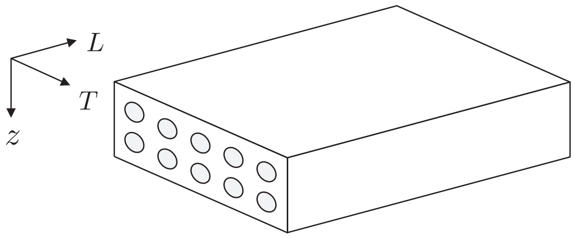
$$\epsilon_{Tu} = \epsilon_{mu}.\left(1 - V_f^{1/3}\right) = \frac{\sigma_{mu}}{E_m}.\left(1 - V_f^{1/3}\right) = 1.537e - 3 \quad [/] \quad . \tag{6}$$

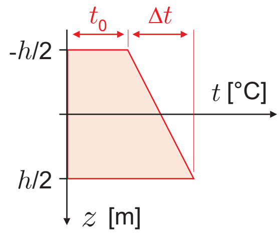
We need to determine the maximum compressive load that can be applied to the composite before it breaks, using the hypotheses $E_L^{'} = E_L$ and $E_T^{'} = E_T$ and the load case $\sigma_L < 0$ and $\sigma_T = \tau_{LT} = 0$. Therefore, we have

$$\sigma_L = E_L.\epsilon_L = -E_L.\frac{\epsilon_T}{\nu_{LT}} \quad \Rightarrow \quad \sigma_{Lu}^{'} = -E_L^{'}.\frac{\epsilon_{Tu}}{\nu_{LT}} = -657.63 \quad [\text{MPa}] \quad , \tag{7}$$

where we have used the definition of the major Poisson's ratio $\epsilon_T = -\nu_{LT}.\epsilon_L$ [/] (explain why we did not use that of the minor Poisson's ratio).

Exercise 2





(a) We make the hypothesis that the distribution of temperatures in the ply follows that depicted in the diagram above : t_0 is the average temperature variation with respect to the reference temperature of the ply t_{ref} (that at which it has been manufactured), and Δt is a (spatial) temperature gradient that varies linearly with the normal coordinate z. The corresponding thermal strain is given by

$$\epsilon = \epsilon(z) = \alpha.(t - t_{ref}) = \alpha.\left(t_0 + \Delta t \,.\, \frac{z}{h}\right) \quad [/] \quad . \tag{8}$$

With respect to the theory of Euler-Bernoulli, the strain at any point of a beam can be written

$$\epsilon = \epsilon(z) = \epsilon_0 - k.z \quad [/] \quad , \tag{9}$$

where ϵ_0 is the average strain and k is the local curvature of the beam, defined by

$$k = \frac{\partial^2 w}{\partial x^2} = w'' \quad [1/\text{m}] \quad , \tag{10}$$

where w is the transverse displacement of the beam along the z axis. Combining eq.(8) to (11), one obtains

$$k = -\frac{\alpha \Delta t}{h} \quad [1/\text{m}] \quad . \tag{11}$$

Approach 1 : going back to the theory of Euler-Bernoulli, the transverse displacements of a beam subjected to pure bending by a torque $M(x)$ follows

$$w'' = -\frac{M}{EI} \quad [1/\text{m}] \quad \Leftrightarrow \quad M = -EIk \quad \Leftrightarrow \quad M = EI.\frac{\alpha \Delta t}{h} \quad [\text{Nm}] \quad . \tag{12}$$

Following the same approach, it is left as an exercise to show that t_0 induces non-zero normal forces in the beam, and that the resulting displacements are in general small with respect to those induced by the bending torque.

Approach 2 : we use the definition of the resultant bending moment over a cross-section of the ply :

$$M = \iint_A z.\sigma dA \quad [\text{Nm}] \quad . \tag{13}$$

which leads to the same result (this is left as a good exercise).

(b) The expression of the displacement field is obtained by integrating twice the curvature of eq.(11) over the length of the beam and applying the boundary conditions (simply supported) to determine the value of the constants. One obtains

$$w(x) = \frac{kx}{2}(x - L) \quad [\mathrm{m}] \quad , \tag{14}$$

which is the equation of a parabola with its maximum a mid-span given by

$$w_{max} = w(x = L/2) = \left| -\frac{kL^2}{8} \right| = \frac{\alpha \Delta t L^2}{8h} \quad [\mathrm{m}] \quad . \tag{15}$$

It is interesting to notice that the displacements are a function of α and not of E. Give a physical explanation.

Using the numerical values given, one finally obtains the results below. Notice how the (tailored) properties of the composites allow changing notably the response of the beam to the same load.

Material	w_{max} [m]
Aluminium	0.074
Kevlar-epoxy	-0.013
Graphite-epoxy	0.0067

Exercise 3

The data allows us to write easily the compliance matrix in the natural axes $\{L, T\}$ using eq.(1). However, the load case is given in the structural axes $\{x, y\}$ and we want to determine the component ϵ_y of the strain tensor. This requires the use of the transformation matrix $[T]$ from the $\{x, y\}$ axes to the $\{L, T\}$ axes (slide 14 of part 3, Section 5.3.4 of the reference book). The steps are described below.

We start by expressing the stresses in the $\{L, T\}$ frame :

$$\begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = [T] \, . \begin{pmatrix} 10e6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7.5 \\ 2.5 \\ 4.33 \end{pmatrix} \quad [\text{MPa}] \quad , \tag{16}$$

Then we use eq.(1) :

$$\begin{pmatrix} \epsilon_L \\ \epsilon_L \\ \gamma_{LT} \end{pmatrix} = [S] \, . \begin{pmatrix} \sigma_L \\ \sigma_T \\ \tau_{LT} \end{pmatrix} = \begin{pmatrix} 67.5e-6 \\ 227.5e-6 \\ 721.7e-6 \end{pmatrix} \quad [/] \quad , \tag{17}$$

Finally, we apply the inverse transformation, using the property $[T(\theta)]^{-1} = [T(-\theta)]$:

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \frac{1}{2}\gamma_{xy} \end{pmatrix} = [T(\theta)]^{-1} \, . \begin{pmatrix} \epsilon_L \\ \epsilon_L \\ \frac{1}{2}\gamma_{LT} \end{pmatrix} = \begin{pmatrix} 420e-6 \\ -125e-6 \\ 499.4e-6 \end{pmatrix} \quad [/] \quad , \tag{18}$$

Hence, we have $\epsilon_y = 125e-6$.

Exercise 4

$[\overline{Q}]$ is the stiffness matrix expressed in the structural axes $\{x, y\}$ (slides 13-16 of part 3, Section 5.3.4 of the reference book) :

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = [\overline{Q}] \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \quad \text{[MPa]} \quad . \tag{19}$$

The components $\overline{Q}_{ij}$ as a function of θ are detailed in the course notes. They have been implemented in Matlab to plot the curves in the figure below. Identify and comment physically on the particular values of θ.