MECA-H-406 Composite structures - Exercises 1: solutions

Exercise 1

The longitudinal Young's modulus, E_L, is given by the **rule of mixtures** :

$$E_L = V_f.E_f + V_m.E_m \quad [Pa] \tag{1}$$

where V_f and V_m are the volume fractions of the fibers and of the matrix, and E_f and E_m are their respective Young's moduli.

The transverse Young's modulus, E_T, can be approximated by the **Halpin-Tsaï model** :

$$E_T = E_m.\frac{1 + \xi\eta V_f}{1 - \eta V_f} \quad [Pa] \quad ; \quad \text{where} \quad \eta = \frac{(E_f/E_m) - 1}{(E_f/E_m) + \xi} \tag{2}$$

where ξ=2 for fibers with a circular cross-section.

Based on this, one obtains the results below.

Material	η [/]	V_f [/]	E_L [GPa]	E_T [GPa]
glass - epoxy	0.864	0.20	16.8	5.7
		0.45	33.4	10.2
		0.60	43.4	14.8
carbon - epoxy	0.971	0.20	72.8	6.0
		0.45	159.4	11.6
		0.60	211.4	18.1
Kevlar - epoxy	0.932	0.20	32.2	5.9
		0.45	68.1	11.1
		0.60	89.6	16.8

Exercise 2

As demonstrated in the course [slides 2-4 of part 2, eq.(3.21) of the reference book], the portion of the total load carried by the fibers is given by

$$\frac{P_f}{P_c} = \frac{E_f/E_m}{E_f/E_m + V_m/V_f} \tag{3}$$

The corresponding curves as a function of V_f are plotted in Fig.1. Because of their high stiffness, the fibers are able to carry a large portion of the total load; in terms of load, the larger E_f/E_m, the more efficient the fibers for a given value of V_f. This, combined to the fact that the fibers generally have ultimate tensile strengths, σ_{fu}, much larger than that of the matrix, allows choosing V_f so as to also increase the ultimate tensile strength of the composite (slide 6 of part 2 - Fig.3.7 of the reference book).

The demonstration of eq.(3) is left as a good exercise.

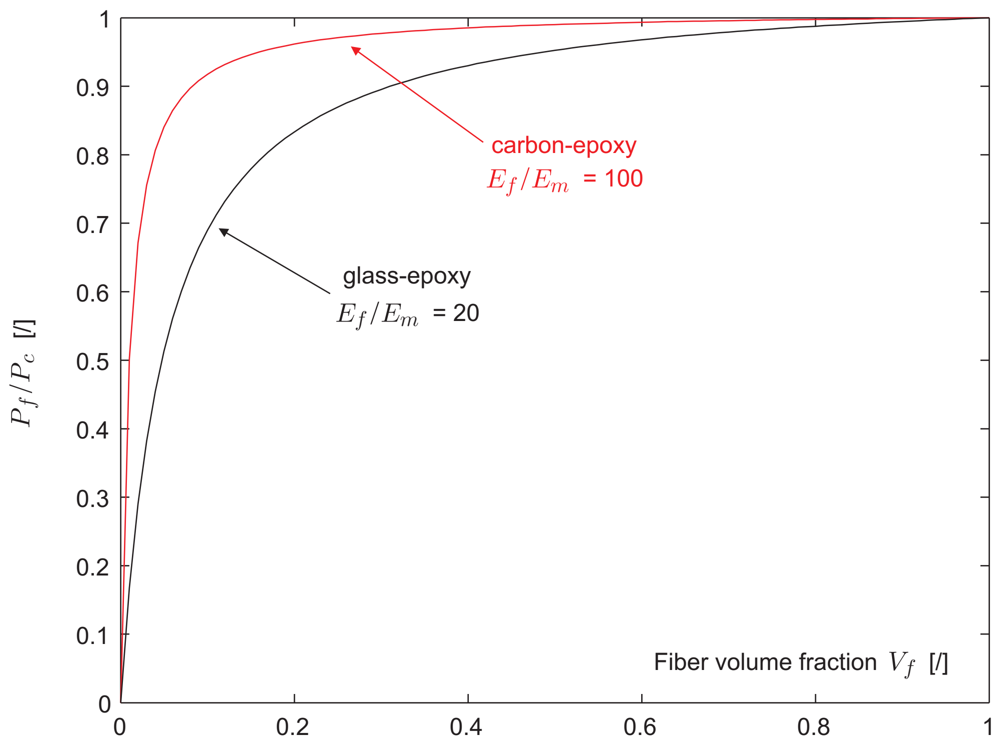

FIGURE 1 – Portion of the total load carried by the fibers for different materials.

Exercise 3

We consider the same load case as in exercise 2. It can been shown that

$$V_{min} = \frac{\sigma_{mu} - (\sigma_m)_{\epsilon_f^*}}{\sigma_{fu} + \sigma_{mu} - (\sigma_m)_{\epsilon_f^*}} \quad [/] \quad \text{and} \quad V_{crit} = \frac{\sigma_{mu} - (\sigma_m)_{\epsilon_f^*}}{\sigma_{fu} - (\sigma_m)_{\epsilon_f^*}} \quad [/] \qquad (4)$$

where σ_{mu} and σ_{fu} are respectively the ultimate tensile stresses of the matrix and of the fiber and $(\sigma_m)_{\epsilon_f^*}$ is the stress carried by the matrix when the fibers break (this implies brittle fibers that break before the matrix).

We start by computing ϵ_f^* for case (a) :

$$\epsilon_{fu} = \frac{\sigma_{fu}}{E_f} = 0.01 \quad , \qquad (5)$$

$$\epsilon_{mu} = \frac{\sigma_{mu}}{E_m} = 0.015 \quad . \qquad (6)$$

Hence, the fibers will break first (i.e. for a smaller strain) so we can use eq.(4) with

$$\epsilon_f^* = 0.01 \quad \Rightarrow \quad (\sigma_m)_{\epsilon_f^*} = E_m.\epsilon_f^* = 35MPa \quad . \qquad (7)$$

The same procedure is applied for the other cases and one obtains the results below.

Material	ϵ_f^* [/]	$(\sigma_m)_{\epsilon_f^*}$ [MPa]	V_{min} [/]	V_{crit} [/]
glass - epoxy	1e-2	35.0	2.4e-2	2.6e-2
carbon - epoxy	2e-3	7.0	6.1e-2	6.6e-2
boron - epoxy	7e-3	25.1	9.7e-3	9.9e-3

Exercise 4

1. Reference : glass fibers - epoxy

We have to compute the ultimate tensile strength of the composite material, σ_{cu}. To do so, we must first determine what will be its behavior with respect to failure as a function of V_f^0 (the exponent 0 denotes the parameters of the reference case), as explained in exercise 3.

Using the results above, in this case again the fibers break first and we have

$$\epsilon_f^* = 0.01 \quad \Rightarrow \quad (\sigma_m)_{\epsilon_f^*} = E_m.\epsilon_f^* = 35MPa \quad . \tag{8}$$

Therefore, using eq.(4), we find

$$V_{min}^0 = 2.44\text{e-}2 \quad \text{and} \quad V_{crit}^0 = 2.6\text{e-}2 \quad , \tag{9}$$

The following condition must be satisfied :

$$V_m^0/V_f^0 = 3 \quad , \tag{10}$$

which, combined with the definition of the volume fractions, $V_f^0 + V_m^0 = 1$, leads to

$$V_f^0 = 0.25 \quad \Rightarrow \quad V_{min}^0 < V_f^0 \quad , \tag{11}$$

hence the matrix cannot support the load alone so σ_{cu} is given by

$$\sigma_{cu}^0 = \sigma_{fu}.V_f^0 + (\sigma_m)_{\epsilon_f^*}.\left(1 - V_f^0\right) = 201.25 \text{ MPa} \quad . \tag{12}$$

2. Carbon & glass fibers - epoxy

We want to determine V_f^C ; the problem is governed by the following equations :

$$V_f^C + V_f^G + V_m = 1 \quad , \tag{13}$$

$$V_m/V_f^G = 3 \quad , \tag{14}$$

$$\sigma_{cu}^1 > \sigma_{cu}^0 \quad . \tag{15}$$

The C and G indices denote the carbon and glass fibers respectively. Again, we have to determine what equation must be used for the computation of σ_{cu}. From the results of exercise 3, we have

$$\epsilon_f^* = 0.002 \quad , \tag{16}$$

meaning that the carbon fibers will break first.

We now choose to consider the glass-epoxy composite as the new matrix material, obeying eq.(14) and with a ultimate tensile strength given by $\sigma^1_{mu} = \sigma^0_{cu} = 201.25$ MPa. We can then compute its longitudinal Young's modulus, E^1_{mL}, and the stress when the carbon fibers break, $\left(\sigma^1_m\right)_{\epsilon^*_f}$:

$$E^1_{mL} = 0.25.E^G_f + 0.75.E_m = 20.125 \text{ GPa} \quad \text{and} \quad \left(\sigma^1_m\right)_{\epsilon^*_f} = E^1_{mL}.\epsilon^*_f = 40.25\text{MPa} \quad . \tag{17}$$

By definition, V_{crit} gives the minimum volume fraction of the (carbon) fibers to be added to the (glass fibers-epoxy) matrix so that the (carbon & glass fibers-epoxy) composite has a higher ultimate strength than the initial (glass fibers-epoxy) matrix. This is achieved for

$$V^C_f \geq V_{crit} = \frac{\sigma^1_{mu} - \left(\sigma^1_m\right)_{\epsilon^*_f}}{\sigma^C_{fu} - \left(\sigma^1_m\right)_{\epsilon^*_f}} = 0.244 \quad . \tag{18}$$

Alternative approach : as V^C_f must ensure condition (15), we have implicitely

$$V^C_f \geq V_{crit} > V_{min} \quad , \tag{19}$$

so the ultimate stress of the newly formed composite is given by

$$\sigma^1_{cu} = \sigma^C_{fu}.V^C_f + \left(\sigma^G_f\right)_{\epsilon^*_f}.V^G_f + \left(\sigma_m\right)_{\epsilon^*_f}.V_m \quad \geq \quad \sigma^0_{cu} \quad , \tag{20}$$

$$\sigma^C_{fu}.V^C_f + \left(E^G_f.\epsilon^*_f\right).\frac{1}{4}\left(1 - V^C_f\right) + \left(E_m.\epsilon^*_f\right).\frac{3}{4}\left(1 - V^C_f\right) \quad \geq \quad \sigma^0_{cu} \quad . \tag{21}$$

where we have used the fact that the carbon fibers break first, the definition of the stresses and eq.(13) and (14). By solving eq.(21), one finds the corresponding value $V^C_f \geq 0.244$.

Exercise 5

(a) Using the rule of mixtures, the longitudinal Young's modulus of the reference glass-epoxy composite is given by

$$E_L^0 = V_f^0.E_f^G + V_m.E_m = 0.65.E_f^G + 0.35.E_m = 46.725 \text{ GPa} \quad . \tag{22}$$

We want to obtain $E_L^1 = 2.E_L^0$ by adding carbon fibers while maintaining $V_m = 0.35$; hence,

$$E_L^1 = V_f^C.E_f^C + V_f^G.E_f^G + V_m.E_m = V_f^C.E_f^C + (0.65 - V_f^C).E_f^G + V_m.E_m = 93.45 \text{ GPa} \quad , \tag{23}$$

which is satisfied for $V_f^C = 0.167$ (i.e. $V_f^G = 0.483$).

(b) The density of each composite is given by the rule of mixtures. We obtain

$$\frac{\rho^1}{\rho^0} = \frac{V_f^C.\rho_f^C + V_f^G.\rho_f^G + V_m.\rho_m}{V_f^0.\rho_f^G + V_m.\rho_m} = 0.943 \quad , \tag{24}$$

hence the new composite is 5.7% lighter than the reference one.

(c) According to exercise 4, for the reference composite, we have

$$\epsilon_f^* = 0.01 \quad \Rightarrow \quad (\sigma_m)_{\epsilon_f^*} = E_m.\epsilon_f^* = 35MPa \quad , \tag{25}$$

and

$$V_f^0 = 0.65 > V_{min}^0 = 2.44e-2 \quad . \tag{26}$$

Therefore, the ultimate tensile strength of the reference glass-epoxy composite is given by

$$\sigma_{cu}^0 = \sigma_{fu}.V_f^0 + (\sigma_m)_{\epsilon_f^*}.\left(1 - V_f^0\right) = 467.25 \text{ MPa} \quad . \tag{27}$$

In the new composite including carbon fibers, as shown in exercise 4, the carbon fibers will break first for $\epsilon_f^* = \epsilon_{fu}^C = 0.002$. Thus if we now consider the glass-epoxy composite as the new matrix material, we can compute its longitudinal Young's modulus, E_{mL}^1, and the stress when the carbon fibers break, $\left(\sigma_m^1\right)_{\epsilon_f^*}$. For this computation, it should be noted that the actual volume fraction of the glass fibers and of the epoxy in the "composite matrix" are given by (0.483/0.833)=0.58 and (0.35/0.833)=0.42 respectively (this differs from exercise 4 where the raio glass/epoxy was a constant).

$$E_{mL}^1 = E_f^G.0.58 + E_m.0.42 = 42.07 \text{ GPa} \quad \text{and} \quad \left(\sigma_m^1\right)_{\epsilon_f^*} = E_{mL}^1.\epsilon_f^* = 84.14\text{MPa} \quad . \tag{28}$$

Similarly, we have to compute the ultimate strength of the composite matrix alone, σ_{mu}^1, considering that the glass fibers break first for $\epsilon_f^* = 0.01$, and that $V_{min}^0 = 2.44e-2 < 0.58$:

$$\sigma_{mu}^1 = \sigma_{fu}.0.58 + (\sigma_m)_{\epsilon_f^*}.0.42 = 420.7\text{MPa} \quad . \tag{29}$$

We can now determine V_{min}^1 :

$$V_{min}^1 = \frac{\sigma_{mu}^1 - \left(\sigma_m^1\right)_{\epsilon_f^*}}{\sigma_{fu}^C + \sigma_{mu}^1 - \left(\sigma_m^1\right)_{\epsilon_f^*}} = 0.325 \quad . \tag{30}$$

So as $V_f^C < V_{min}^1$, the addition of the carbon fibers actually weakens the new composite and

$$\sigma_{cu}^1 = \sigma_{mu}^1.(1 - V_f^C) = 350.4\text{MPa} \quad . \tag{31}$$

However, the condition $V_f^C < V_{min}$ also means that the glass-epoxy "matrix" is strong enough to support the load when the carbon fibers break, so there is still a safety margin with respect to the total failure of the material (breaking of the glass fibers).