Unidirectional composite

Volume fractions:

\[ V_f = \frac{v_f}{v_c}, \quad V_m = \frac{v_m}{v_c} \]

0.3 < V_f < 0.8

Density:

\[ \rho_c v_c = \rho_f v_f + \rho_m v_m \]

\[ \rho_c = \rho_f \frac{v_f}{v_c} + \rho_m \frac{v_m}{v_c} \]

\[ \rho_c = \rho_f V_f + \rho_m V_m \]
1. Longitudinal strength and stiffness

1. The fibers and the matrix experience the same strain

\[ \epsilon_f = \epsilon_m = \epsilon_c \]

2. The load is shared between the fibers and the matrix

\[ P_c = P_f + P_m \]

\[ P_c = \sigma_c A_c = \sigma_f A_f + \sigma_m A_m \]

\[ \sigma_c = \sigma_f \frac{A_f}{A_c} + \sigma_m \frac{A_m}{A_c} \]

\[ \sigma_c = \sigma_f V_f + \sigma_m V_m \]

\[ \frac{d\sigma_c}{d\epsilon} = \frac{d\sigma_f}{d\epsilon} V_f + \frac{d\sigma_m}{d\epsilon} V_m \]

**Rule of mixtures:**

\[ E_c = E_f V_f + E_m V_m \]

\[ E_c = \sum_{i=1}^{n} E_i V_i \]

[Good agreement with experiments in traction
Some deviation in compression due to buckling of the fibers]
\[ \frac{E_c}{E_m} = \left( \frac{E_f}{E_m} - 1 \right) V_f + 1 \]

\[ \frac{E_c}{E_f} = V_f + \frac{E_m}{E_f} V_m \]

Example: \( E_{\text{glass}} = 70 \text{ Gpa}, E_{\text{carbon}} = 350 \text{ Gpa}, E_{\text{epoxy}} = 3.5 \text{ GPa} \)

<table>
<thead>
<tr>
<th>System ((E_f/E_m))</th>
<th>(E_c/E_m) (V_f = 10%)</th>
<th>(E_c/E_m) (V_f = 50%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glass–epoxy (20)</td>
<td>2.9</td>
<td>10.5</td>
</tr>
<tr>
<td>Carbon–epoxy (100)</td>
<td>10.9</td>
<td>50.5</td>
</tr>
</tbody>
</table>

The properties in the longitudinal direction are dominated by the fibers

\( \sigma \) vs. \( \varepsilon \)

Figure 3.4. Longitudinal stress–strain diagrams for hypothetical composites with (a) linear and (b) nonlinear matrix material.
Fraction of the load carried by the fibers:

\[
\frac{P_f}{P_c} = \frac{\frac{E_f \varepsilon_f A_f}{E_f \varepsilon_f A_f + E_m \varepsilon_m A_m}}{\frac{E_f V_f}{E_f V_f + E_m V_m}} = \frac{E_f E_m}{E_f E_m + V_m V_f}
\]

- Carbon/epoxy: \(E_f/E_m = 100\)
- Glass/epoxy: \(E_f/E_m = \) values provided

- Glass/epoxy: \(0.69, 0.952\)
- Carbon/epoxy: \(0.917, 0.99\)

Figure 3.5. Load carried by the fibers as the percentage of total load on the composite.
Stress-strain curve of an hypothetical composite with **ductile** and **brittle** fibers and typical ductile matrix

- The composites with brittle fibers fracture at the fracture strain of the fibers.
- If the fibers are ductile, the fracture strain of the fibers in the composite may be larger than without the matrix.
**Ultimate strength of unidirectional composite as function of fiber volume fraction**

\[ V_{\text{min}} = \text{minimum fiber volume fraction} \]

that ensures fiber-controlled composite failure

\[ V_{\text{min}} = \frac{\sigma_{\text{mu}} - (\sigma_m) \epsilon_f^*}{\sigma_{\text{fu}} + \sigma_{\text{mu}} - (\sigma_m) \epsilon_f^*} \]

**\[ V_{\text{crit}} = \text{critical volume fraction} \]**

above which the strength of the composite exceeds that of the matrix

\[ V_{\text{crit}} = \frac{\sigma_{\text{mu}} - (\sigma_m) \epsilon_f^*}{\sigma_{\text{fu}} - (\sigma_m) \epsilon_f^*} \]

Glass/epoxy:

- \( \sigma_{\text{fu}} = 2500 \text{ Mpa} \)
- \( \epsilon_{\text{fu}} = 3.5\% \)
- \( \sigma_{\text{mu}} = 130 \text{MPa} \)
- \( E_m = 3500 \text{MPa} \)

\( V_{\text{crit}} < 1\% \)
2. Transverse stiffness and strength

2.1. Constant stress model:

The composite is stressed in the transverse direction, perpendicular to the fibers. It is represented as made by layers of fibers and layers of matrix, with the same area, and experiencing the same load and the same stress.

Elongations:

\[ \delta_c = \epsilon_c t_c \]
\[ \delta_f = \epsilon_f t_f \]
\[ \delta_m = \epsilon_m t_m \]

Assumption!!

Inverse rule of mixtures

Problems:
1. The assumption of uniform stress is wrong
2. Mismatch of Poisson’s ratios \( v_f \) and \( v_m \)
Figure 3.10. Numerical predictions of composite transverse modulus. Source: Adams and Doner [26].
Inverse rule of mixtures:

\[ E_T = E_m \left[ \frac{1}{(1 - V_f) + (E_m/E_f)V_f} \right] \]

2.2. Halpin-Tsai semi-empirical model

\[ \frac{E_T}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \]

\[ \eta = \frac{(E_f/E_m) - 1}{(E_f/E_m) + \xi} \]

\( \xi \) is obtained by curve fitting on elasticity solutions

For circular or square fibers: \( \xi = 2 \)

For rectangular fibers: \( \xi = 2a/b \) (a in the loading direction)

(For \( \xi = 0 \), it reduces to the inverse rule of mixtures)
Example:

\[
\eta = \frac{(E_f/E_m) - 1}{(E_f/E_m) + \xi}
\]

\[
\frac{E_T}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f}
\]

(circular fibers: \(\xi = 2\))

### Glass–Epoxy System

\[
\frac{E_f}{E_m} = 20 , \quad \eta = \frac{20 - 1}{20 + 2} = \frac{19}{22}
\]

\[
V_f = 10\% , \quad \frac{E_T}{E_m} = \frac{1 + 2 \times \frac{19}{22} \times 0.1}{1 - \frac{19}{22} \times 0.1} = 1.28
\]

\[
V_f = 50\% , \quad \frac{E_T}{E_m} = \frac{1 + 2 \times \frac{19}{22} \times 0.5}{1 - \frac{19}{22} \times 0.5} = 3.28
\]

### Carbon–Epoxy System

\[
\frac{E_f}{E_m} = 100 , \quad \eta = \frac{100 - 1}{100 + 2} = \frac{99}{102}
\]

\[
V_f = 10\% , \quad \frac{E_T}{E_m} = \frac{1 + 2 \times \frac{99}{102} \times 0.1}{1 - \frac{99}{102} \times 0.1} = 1.32
\]

\[
V_f = 50\% , \quad \frac{E_T}{E_m} = \frac{1 + 2 \times \frac{99}{102} \times 0.5}{1 - \frac{99}{102} \times 0.5} = 3.83
\]

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<th>System ((E_f/E_m))</th>
<th>(V_f = 10%)</th>
<th>(V_f = 50%)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>(E_T/E_m)</td>
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</tr>
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<td>1.28</td>
</tr>
<tr>
<td>Carbon–epoxy (100)</td>
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<td>1.32</td>
</tr>
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Almost independent of \(E_f/E_m\)
2.3. Transverse strength

- In the longitudinal direction, the load is shared by the fibers and the matrix.
- In the transverse direction, the fibers are unable to take a large part of the load.
- The high modulus fibers serve as effective constraints on the deformation of the matrix.
- The fibers are responsible for stress concentration.

The composite failure occurs at a much lower strain than the unrestrained matrix material.

Composite breaking strain:

\[ \epsilon_{cB} = \epsilon_{mB} \left( 1 - V_f^{1/3} \right) \]

Stress concentration in matrix surrounding a single cylindrical inclusion

\[ E_f/E_m=10, \ \nu_m=0.35, \ \nu_f=0.3 \]
Normalized principal stress in matrix surrounding multiple fibers

\[ \nu_m = 0.35 \quad \nu_f = 0.20 \]
3. Shear modulus $G_{LT}$

3.1 Constant stress model

\[ \Delta_c = \gamma_c t_c \]
\[ \Delta_f = \gamma_f t_f \]
\[ \Delta_m = \gamma_m t_m \]

\[ \gamma_c t_c = \gamma_f t_f + \gamma_m t_m \quad \rightarrow \quad \gamma_c = \gamma_f \frac{t_f}{t_c} + \gamma_m \frac{t_m}{t_c} \]
\[ = \gamma_f V_f + \gamma_m V_m \]

\[ \frac{1}{G_{LT}} = \frac{V_f}{G_f} + \frac{V_m}{G_m} \]

\[ \frac{\tau_c}{G_{LT}} = \frac{\tau_f}{G_f} V_f + \frac{\tau_m}{G_m} V_m \]

Inverse rule of mixtures
3.2. Halpin-Tsai semi-empirical model

\[ \frac{G_{LT}}{G_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \]

\[ \eta = \frac{(G_f/G_m) - 1}{(G_f/G_m) + \xi} \]

Same as for \( E_T \), except that \( \xi = 1 \)

*Figure 3.14. Numerical predictions of composite shear modulus. Source: Adams and Doner [26].*
4. Prediction of the Poisson’s ratio

4.1. Major $\nu_{LT}$

$$(\varepsilon_T)_f = -\nu_f(\varepsilon_L)_f$$

$$(\varepsilon_T)_m = -\nu_m(\varepsilon_L)_m$$

$$(\varepsilon_T)_c = -\nu_{LT}(\varepsilon_L)_c$$

$\delta_f = t_f(\varepsilon_T)_f = -t_f\nu_f(\varepsilon_L)_f$

$\delta_m = t_m(\varepsilon_T)_m = -t_m\nu_m(\varepsilon_L)_m$

$\delta_c = t_c(\varepsilon_T)_c = -t_c\nu_{LT}(\varepsilon_L)_c$

![Figure 3.16. Model of unidirectional composite for prediction of Poisson’s ratio.](image)

The deformation of the composite is the sum of that of the matrix and of the fiber:

$-t_c\nu_{LT}(\varepsilon_L)_c = -t_f\nu_f(\varepsilon_L)_f - t_m\nu_m(\varepsilon_L)_m$

$t_c\nu_{LT} = t_f\nu_f + t_m\nu_m$

$\nu_{LT} = \nu_f V_f + \nu_m V_m$

Rule of mixtures
4.2. Minor $v_{TL}$

Load case No 1: $\sigma_L = 1$

- $\sigma_L = 1$
- $\varepsilon_L = 1/E_L$
- $\varepsilon_T = -\nu_{LT}/E_L$

Load case No 2: $\sigma_T = 1$

- $\sigma_T = 1$
- $\varepsilon_L = -\nu_{TL}/E_T$
- $\varepsilon_T = 1/E_T$

According to Maxwell's Principle of Reciprocity,

$$\frac{\nu_{LT}}{E_L} = \frac{\nu_{TL}}{E_T}$$
Table 3.2 Summary of influence of constituents on properties of unidirectional polymer composites

<table>
<thead>
<tr>
<th>Composite property</th>
<th>Fibers</th>
<th>Matrix</th>
<th>Interface</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tensile Properties</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Longitudinal modulus</td>
<td>S</td>
<td>W</td>
<td>N</td>
</tr>
<tr>
<td>Longitudinal strength</td>
<td>S</td>
<td>W</td>
<td>N</td>
</tr>
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<td>W</td>
<td>S</td>
<td>N</td>
</tr>
<tr>
<td>Transverse strength</td>
<td>W</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td><strong>Compression Properties</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Longitudinal modulus</td>
<td>S</td>
<td>W</td>
<td>N</td>
</tr>
<tr>
<td>Longitudinal strength</td>
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<td>S</td>
<td>N</td>
</tr>
<tr>
<td>Transverse modulus</td>
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<td>N</td>
</tr>
<tr>
<td>Transverse strength</td>
<td>W</td>
<td>S</td>
<td>N</td>
</tr>
<tr>
<td><strong>Shear Properties</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>In-plane shear modulus</td>
<td>W</td>
<td>S</td>
<td>N</td>
</tr>
<tr>
<td>In-plane shear strength</td>
<td>W</td>
<td>S</td>
<td>S</td>
</tr>
<tr>
<td>Interlaminar shear strength</td>
<td>N</td>
<td>S</td>
<td>S</td>
</tr>
</tbody>
</table>

*S = strong influence; W = weak influence; N = negligible influence.*