Finite element modelling of smart piezoelectric shell structures

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1 Introduction

The design of control systems involving piezoelectric actuators and sensors requires an accurate knowledge of the transfer functions between the inputs and the outputs of the system. [6] These are not easy to determine numerically, particularly for shell structures with embedded distributed actuators and sensors. This paper presents the finite element modeling of smart piezolaminated shells structures. The fundamental equations governing the equivalent piezoelectric loads and sensor output are derived. The reciprocity between piezoactuation and piezosensing is pointed out. The finite element formulation is presented. The interfacing with a control oriented software is discussed and an application in vibroacoustics is presented.

2 Constitutive equations

We consider a shell structure with embedded piezoelectric patches covered with electrodes. The piezoelectric patches are parallel to the mid-plane and orthotropic in their plane. The electric field and electric displacement are assumed uniform across the thickness and aligned on the normal to the mid-plane. With the plane stress hypothesis, the constitutive equations of a linear piezoelectric material read [4].

\[ T = c^E S - e^T E \]

\[ D = e S + \varepsilon S E \]

where \( T = \{T_{11}, T_{22}, T_{12}\}^T \) is the stress vector, \( S = \{S_{11}, S_{22}, 2S_{12}\}^T \) the deformation vector, \( E \) the electric field, \( D \) the electric displacement, \( \varepsilon \) the dielectric constant at \( S \) constant, \( c^E \) the stiffness matrix of the piezoelectric material in its orthotropy axes at \( E \) constant, and \( e = \{e_{31}, e_{32}, 0\} \) the piezoelectric constants.

A laminate is formed from several layers bonded together to act as a single layer material (Fig.1); the bond between two layers is assumed to be perfect, so that the displacements remain continuous across the bond.

\[ S = S_0 + z \kappa \]

According to the Kirchhoff hypothesis, a fiber normal to the mid-plane remains so after deformation. It follows that:

\[ \{N \ M\} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \{S_0 \ \kappa\} + \sum_{k=1}^{n} I_3 \begin{bmatrix} z_{mk} \ I_3 \end{bmatrix} R_{T_k}^{-1} e_k \phi_k \]

where \( R_{T_k}^{-1} \) is the transformation matrix relating the stresses in the local coordinate system \((LT)\) to the global one \((xy)\) for layer \( k \), \( z_{mk} \) is the distance from the mid-plane of layer \( k \) to the mid-plane of the laminate. The first term in the right hand side of Eqn.(4) is the classical stiffness matrix of a composite laminate, where the extensional stiffness matrix \( A \), the bending stiffness matrix \( D \) and the extension/bending coupling matrix \( B \) are
related to the individual layers according to the classical relationships that can be found in any textbook on composite materials. [1] The second term in the right hand side of Equ.(4) expresses the piezoelectric loading.

Since the electric displacement $D_k$ is assumed uniform over the thickness of layer $k$, Equ.(2) can be averaged over the thickness. After substituting Equ.(3), one gets

$$D_k = e_k R_{Sk} |I_3 z_{mk} I_3| \left\{ S_0 \kappa \right\} - \frac{e_k}{h_k} \phi_k$$

where $R_{Sk}$ is the transformation matrix relating the strains in the global coordinate system ($xy$) to the local one ($LT$) for layer $k$.

3 Actuation

If the piezoelectric properties are isotropic in the plane ($e_{31} = e_{32}$), we have $R_{Tk}^{-1} e_k = e_k$. The in-plane forces and the bending moments are both hydrostatic; they are independant of the orientation of the facet. The piezoelectric loads result in a uniform in-plane load $N_p$ and bending moment $M_p$ acting normally to the contour of the electrode as indicated on Fig.2:

$$N_p = -e_{31} \phi, \quad M_p = -e_{31} z_m \phi$$

where $z_m$ is the distance from the mid-plane of the piezoelectric patch to the mid-plane of the plate.

4 Sensing

Consider a piezoelectric patch connected to a charge amplifier as in Fig.3. The charge amplifier imposes $\phi = 0$ between the electrodes and the output voltage is proportional to the electric charge:

$$\phi_{out} = -\frac{Q}{C_r} = -\frac{1}{C_r} \int \Omega D \, d\Omega$$

where $D$ is given by Equ.(5). If the piezoelectric properties are isotropic in the plane ($e_{31} = e_{32}$), we have $e_k R_{Sk} = e_k$ and Equ.(7) becomes

$$\phi_{out} = -\frac{e_{31}}{C_r} \left[ \int_\Omega S_0^x + S_0^y \, d\Omega \right] + z_m \int_\Omega (\kappa_x + \kappa_y) \, d\Omega$$

The first integral represents the contribution of the average membrane strains over the electrode and the second, the contribution of the average bending moment. Using the Green integral ($\int_\Omega \nabla \cdot \vec{a} \, d\Omega = \int_C \vec{a} \cdot \vec{n} \, dl$) the foregoing result can be transformed into

$$\phi_{out} = -\frac{e_{31}}{C_r} \left[ \int_C \bar{u}_n \vec{n} \, dl + z_m \int_C \frac{\partial w}{\partial n} \, dl \right]$$

where the integrals extend to the contour of the electrode. The first term is the mid-plane displacement normal to the contour while the second is the slope of the mid-plane in the plane normal to the contour (Fig.4). It is worth insisting that for both the actuator and the sensor, it is not the shape of the piezoelectric patch that matters, but rather the shape of the electrodes.

5 Finite element formulation

The dynamic equations of a piezoelectric continuum can be derived from the Hamilton principle, in which the potential energy density and the
virtual work are properly adapted to include the electrical contributions as well as the mechanical ones. [7].

The elements used are the Mindlin shell elements from Samcef (Samtech s.a.). The electrical degrees of freedom are the voltages $\phi_k$ across the piezoelectric layers; it is assumed that the potential is constant over each element (this implies that the finite element mesh follows the shape of the electrodes). Introducing the nodal displacements $q$, we get

\begin{align}
M \ddot{q} + K_{qq} q + K_{\phi q} \phi &= f \\
K_{\phi q} q + K_{\phi \phi} \phi &= g
\end{align}

where $f$ and $g$ are respectively the external forces reduced to the mechanical degrees of freedom and electric charges brought to the electrical degrees of freedom. The expression of the element mass $M$, stiffness $K_{qq}$, coupling $K_{\phi q}=K_{\phi q}^T$ and piezoelectric capacitance $K_{\phi \phi}$ matrices can be found in [5].

The element coordinates $q$ and $\phi$ are related to the global coordinates $Q$ and $\Phi$. The assembly takes into account the equipotentiality condition of the electrodes; this reduces the number of electric variables to the number of electrodes. We get the global system of equations

\begin{align}
M_{QQ} \ddot{Q} + K_{QQ} Q + K_{Q \Phi} \Phi &= F \\
K_{Q \Phi} Q + K_{\Phi \Phi} \Phi &= G
\end{align}

where the global matrices can be derived in a straightforward manner from the element matrices. As for the element matrices, the global coupling matrices satisfy $K_{\Phi Q}=K_{\Phi Q}^T$.

\section{State space model}

Equations (12) and (13) can be complemented with a damping term $C\dot{Q}$ to obtain the full equation of dynamics and the sensor equation:

\begin{align}
M \ddot{Q} + C \dot{Q} + K_{QQ} Q + K_{Q \Phi}^{(i)} \Phi &= 0 \\
G &= K_{Q \Phi}^{(o)} Q + K_{\Phi \Phi} \Phi
\end{align}

Actuation is done by imposing a voltage $\Phi$ on the actuators and sensing by imposing $\Phi = 0$ and measuring the electric charges $G$ appearing on the sensors.

Using the modal decomposition $Q = Z \ x(t)$, where $Z$ represents the modal shapes ($n$ decoupled modes) and $x(t)$ the modal amplitudes, and left -multiplying by $Z^T$, Equ.(14) becomes

\begin{align}
0 &= Z^T M Z \ddot{x} + Z^T C Z \dot{x} + Z^T K_{QQ} Z x + Z^T K_{Q \Phi}^{(i)} \Phi
\end{align}

and finally, the dynamic equations of the system in the state space representation read:

\begin{align}
\begin{bmatrix}
\dot{x} \\
\dot{\phi}
\end{bmatrix} &=
\begin{bmatrix}
0 & -\Omega^2 \\
-2\xi \Omega & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\dot{x}
\end{bmatrix}
- \begin{bmatrix}
0 \\
\mu^{-1} Z^T K_{Q \Phi}^{(i)}
\end{bmatrix} \Phi \\
G &= [K_{Q \Phi}^{(o)} Z \ 0] \begin{bmatrix}
\dot{x} \\
\dot{\phi}
\end{bmatrix} + D_{HF} \Phi
\end{align}

where the modal frequencies $\Omega = \text{diag}(\omega_k)$, the modal shapes $Z$, the modal masses $\mu = \text{diag}(\mu_k)$, the modal electric charge on the sensor $K_{Q \Phi}^{(o)} Z$, and the modal electric charge on the actuator, transposed (by reciprocity) $Z^T K_{Q \Phi}^{(i)}$ are obtained from the FE dynamic analysis. $\xi$ represents the modal damping ratios of the considered structure and $D_{HF}$ is the static contribution of the non-modelized high frequency mode; Its elements are given by

\begin{align}
D_{lm} = & \left( K_{Q \Phi}^{(o)} Z l \right) (Z^T K_{Q \Phi}^{(i)})_m \\
& \sum_{k=1}^{n} \frac{(Z^T K_{Q \Phi}^{(i)})_m}{\mu_k \omega_k^2}
\end{align}

where $d_{lm}$ is the charge appearing on sensor $l$ when a unit voltage is applied on actuator $m$ and is obtained from a static FE analysis.

Such a state space representation is easily implemented in a control oriented software allowing the designer to extract the various transfer functions and use the control design tools.

\section{ASAC plate}

The ASAC plate (Active Structural Acoustical Control) [3] is a volume velocity control device. It consists in a clamped plate of aluminium covered on both side with piezoelectric PVDF film (Fig.5). It exhibits a pair of colocated quadratically shaped actuator/sensor. For the actual

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ASAC_setup.png}
\caption{ASAC experimental setup}
\end{figure}
laboratory model (KUL-PMA) the actuation and sensing layers electrodes present 24 strips (Fig.5). The direction of smaller piezoelectric coupling coefficient \( e_{32} \) is perpendicular to the strips. The finite element mesh used is shown on Fig.6.

In a first attempt to modelize this control device, the agreement with experiments was rather poor as shown on Fig.7 (FE Result #1).

Further investigations shown that the difference comes from the clamping characteristics. In the actual experiment, the in-plane movement of the plate is not strictly fixed by the clamp as assumed in the first FE simulations.

Freing the in-plane movement of the plate results in a stronger influence of the membrane component and, therefore, in a stronger in-plane mechanical coupling between actuator and sensor. This induces an important feedthrough term in the transfer function. The results of the FE simulation with the correct boundary conditions are also shown on Fig.7 (FE Result #2) and show a very good agreement with the experiments.

This test case illustrates the situation of shell structures with embedded piezoelectric actuators and sensors where they are nearly collocated. It stresses the importance of membrane components on the zeros of the transfer function. These local effects can easily be accounted for by the developed modelling tools based on finite elements.

8 Conclusions

The finite element modeling of smart piezolaminated shells structures has been presented; the fundamental equations governing the equivalent piezoelectric loads of a piezoelectric actuator and the output of a piezoelectric sensor have been derived. A state space model has been obtained and an application in vibroacoustics stressing the importance of the in-plane components in the open loop transfer functions has been presented.

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10 References