TOOLS FOR A MULTIAXIAL FATIGUE ANALYSIS OF STRUCTURES SUBMITTED TO RANDOM VIBRATIONS

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ABSTRACT

This paper proposes a frequency domain method to estimate the high-cycle fatigue damage for multiaxial stresses caused by random vibration, directly from a spectral analysis. This approach is based on a new definition of the von Mises stress as a random process. It is shown that the approach can be generalized to include a frequency domain formulation of the multiaxial rainflow method for biaxial stress states. The results obtained with both methods are compared. Finally, this paper describes a procedure for the generation of multiaxial stress tensor histories of a given PSD matrix.

Key words: multiaxial high cycle fatigue, random vibration.

1. INTRODUCTION

High-cycle fatigue due to multiaxial random loads is still the subject of ongoing basic research. Some multiaxial damage models have been proposed during the last decade and are based on stress, strain or energy. For these methods, the full local stress or/and strain tensors are required. Such methods would be very costly and time consuming to apply to each finite element of a finely discretized structure, and impossible to apply in an early design stage. This paper proposes a numerical procedure to estimate the high-cycle fatigue damage for multiaxial random stresses, directly from a spectral analysis.

The originality of the approach is that the equivalent alternating uniaxial stress is constructed in the frequency domain by combining, for each frequency, the power spectral densities of the normal and shear stresses according to the quadratic von Mises rule. The power spectral density of the equivalent process is thus obtained. Consequently, all the uniaxial fatigue assessment methods available can be applied in the time domain (rainflow counting) as well as in the frequency domain (spectral moment methods).

In a second part of this paper, a formulation of the multiaxial rainflow method in the frequency domain is proposed. The PSD of all linear combinations of the stress tensor components is computed directly from the PSD-matrix. A set of uniaxial processes is thus obtained and the most damaging one is found.

Both methods applied in the frequency domain supply a complete map of the fatigue damage in every element and are in excellent agreement.

The objective of the previous approaches is not to predict fatigue life very accurately, but are rather to be used to compare design alternatives and to identify the most critical locations in a structure submitted to random loads. The designer may then wish to analyse further the critical elements. For this purpose a multiaxial time history generation tool has been developed and a procedure to generate a set of time histories of the stress tensor that fit a given PSD matrix is described. These stress histories can later be used in a more accurate multiaxial fatigue life assessment method based for example on the critical plane approach.

2. UNIAXIAL GAUSSIAN STRESS

The equivalent von Mises stress approach is based on extensions of static yield theory and on a new definition of the von Mises stress as a random process. It assumes that fatigue damage under multiaxial loading can be predicted by calculating an equivalent uniaxial stress or strain on which the classical uniaxial random fatigue theory is applied.

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Rayleigh Approximation

Consider a material characterized by a deterministic S-N curve of the form $NS^\beta = c$, where S is the constant amplitude of the alternating sinusoidal stress and N is the number of cycles to failure; the constant c and β are characteritics of the material ($5 < \beta < 20$). We assume that the linear damage theory apply.

According to the classical theory of random fatigue, every positive maximum of amplitude a contributes for one cycle to the damage, that is:

$$D = c^{-1}a^\beta \quad (1)$$

The contribution to the damage rate from the infinitesimal range $[a, a+da[$ is given by:

$$\Delta(a)da = c^{-1}a^\beta \nu_0 p(a)da \quad (2)$$

where $\nu_0 p(a)da$ is the average number of cycles in the range $[a, a+da[$. For a narrow-band process, $p(a)$ is the Rayleigh distribution and the rate of maxima can be approximated by the central frequency ν_0. Integrating from 0 to infinity, we get the well known result :

$$\mathrm{E}[D] = \int_0^\infty \Delta(a)da = c^{-1}\sigma_x^\beta \nu_0 2^{\beta/2}\Gamma(1+\beta/2) \quad (3)$$

where σ_x is the RMS value of the stress and $\Gamma(.)$ is the Gamma function. Equation (3) can be written alternatively in terms of the spectral moments as

$$\mathrm{E}[D] = c^{-1}\frac{2^{\beta/2}}{2\pi}\Gamma(1+\beta/2)m_0^{(\beta-1)/2}m_2^{1/2} \quad (4)$$

where the spectral moments are defined as

$$m_a = 2\int_0^\infty \omega^a \Phi(\omega)d\omega \quad (5)$$

and $\Phi(\omega)$ is the double-sided PSD function. This result can also be applied as an approximation for a wide-band process (P.H.Wirshing and E.B.Haugen, 1973; S.H.Crandall and W.D.Mark, 1963).

Single Moment :

The following heuristic formula has been proposed by Larsen and Lutes (L.D.Lutes and C.E.Larsen, 1990) as an alternative to Equ.(4):

$$\mathrm{E}[D] = c^{-1}\frac{2^{\beta/2}}{2\pi}\Gamma(1+\beta/2)(m_{2/\beta})^{\beta/2} \quad (6)$$

It takes its name from the fact that it depends only on the spectral moment $m_{2/\beta}$. Although there is no theoritical base to Equ.(6), it gives the correct dependence on both ω and σ_x and is equivalent to the Rayleigh approximation for narrow-band processes. The single moment method is in closer agreement with rainflow simulations for various spectral shapes, including for bimodal PSD.

Rainflow cycle counting :

In high-cycle uniaxial fatigue, the most accurate variable for correlating damage is the stress amplitude of closed hysteresis loops of the local stress-strain path at the crack location site. In the elastic domain these loops are flat. The proper way of extracting such cycles is the rainflow counting method (N.E.Dowling, 1972). This counting is applied after Monte Carlo simulations of stress time histories of a given spectral content.

Definition : Let $\sigma(t)$, $0 \le t \le T$ be a stress and denote by M_i the peaks of $\sigma(t)$ at time t_i, $t_i \le t_{i+1}$ (see Figure 1). To define the rainflow cycles, each local maximum M_i has to be paired with one particular local minimum m_i, found as follows : from a local maximum M_i with height u, say, one shall try to reach above u (in the forward and backward direction) with an as small downward excusion as possible. The minimum $m_i^+ = m_i^{RFC}$, which represents the smallest deviation form the maximum M_i, is defined to be the corresponding rainflow minimum. Thus the rainflow cycle starting at M_i is (m_i^{RFC}, M_i) (I.Rychlik, 1987).

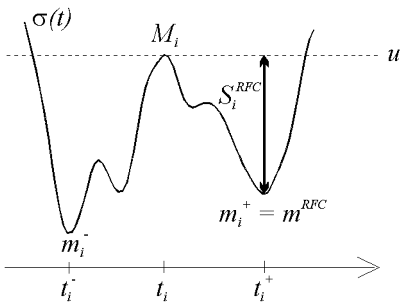

Figure 1: Rainflow cycle

Damage estimation : From this counting, each stress cycle (σ_m, σ_a), where σ_m is the mean stress and σ_a is the stress amplitude, is tranformed into an equivalent cycle of zero mean $(0, \sigma_{a_{eq}})$ according to the Goodmann diagram. Afterward, the damage corresponding to each cycle is computed according to the S-N curve, and the linear cumulative Palmgren-Miner law is applied.

3. EQUIVALENT VON MISES STRESS

Fatigue cracks normally initiate at free surfaces, where the stress state is biaxial. For such multiaxial stress fields, the fatigue phenomenon is generally regarded as being governed by a combination of the shear and normal stress acting on a critical plane. It has also been reported in the literature (G.Sines and G.Ohgi, 1987) that the von Mises criterion correlates fairly well with a large amount of experimental data for biaxial stress states with constant principal directions. When the excitation is random, the principal directions can rotate continuously with time ; the von Mises stress is proportional to the root mean square of the shear stress over all planes.

In this study, we propose to base a first estimate of the fatigue life on the von Mises stress which is used as an equivalent uniaxial counting variable (A.Preumont, 1994; A.Preumont and V.Piéfort, 1994).

For a biaxial stress, the von Mises stress s_c is defined by the quadratic relationship

$$s_c^2 = s_x^2 + s_y^2 - s_x s_y + 3s_{xy}^2 \tag{7}$$

where s_x, s_y and s_{xy} are the normal and tangential stresses, respectively. Defining the stress vector as $s = (s_x, s_y, s_{xy})^T$, Equ.(7) can be rewritten

$$s_c^2 = s^T Q s = \text{Trace}\{Q[ss^T]\} \tag{8}$$

with

$$Q = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{9}$$

After taking the expectation,

$$\text{E}[s_c^2] = \text{Trace}\{Q\text{E}[ss^T]\} \tag{10}$$

where $E[SS^T]$ is the covariance matrix of the stress vector, related to the PSD matrix of the stress vector by

$$E[SS^T] = \int_{-\infty}^{\infty} \Phi_{ss}(\omega) d\omega \tag{11}$$

From these equations, we can *define* the PSD $\Phi_c(\omega)$ of the *equivalent von Mises stress* as a frequency decomposition of its mean square value:

$$\begin{aligned} \text{E}[s_c^2] &= \int_{-\infty}^{\infty} \Phi_c(\omega) d\omega \\ &= \int_{-\infty}^{\infty} \text{Trace}\{Q\Phi_{ss}(\omega)\} d\omega \end{aligned} \tag{12}$$

where $\Phi_{ss}(\omega)$ is the PSD matrix of the stress vector. Equivalently,

$$\Phi_c(\omega) = \text{Trace}\{Q\Phi_{ss}(\omega)\} = \sum_{i,j} Q_{ij} \Phi_{s_i s_j}(\omega) \tag{13}$$

Note that, in the uniaxial case [where only $s_x \neq 0$, with a PSD $\Phi_{Sx}(\omega)$], Equ.(13) supplies $\Phi_c(\omega) = \Phi_{Sx}(\omega)$. Equation (13) defines the *equivalent alternating stress (von Mises stress) as the scalar random process whose* PSD *is obtained from the* PSD *matrix of the stress components according to the von Mises quadratic combination rule.* The von Mises stress is then considered as an equivalent counting variable taking into account the multiaxiality. It is this scalar process that we propose to use in the subsequent fatigue damage analysis according to a uniaxial prediction model for Gaussian processes such as the Rayleigh approximation, the Single Moment or rainflow simulations. The methodology developed for biaxial stress states can readily be extended to triaxial stress states, it is formally the same, with different definitions of the stress vector and the Q matrix. A finite element formulation of the foregoing method is presented in (A.Preumont and V.Piéfort, 1994).

4. FREQUENCY DOMAIN FORMULATION OF THE MULTIAXIAL RAINFLOW METHOD

4.1. Multiaxial Rainflow Counting

Let $Y(t)$ be a random n-dimensional vector process ; the n signals may be thought of either as load components of external forces acting on the structure or as components of the local stress or strain tensor at a given point. The idea is to count rainflow cycles on all linear combinations $Y_c(t)$ of the random vector components of the form :

$$Y_c(t) = \sum_{i=1}^{n} c_i Y_i(t)$$

where c_i belong to a hypersphere $\sum_{i=1}^{n} c_i^2 = 1$. Practically, when the stress state is biaxial, the stress components can be written under the form of the three dimension vector $s = (s_x, s_y, s_{xy})^T$. A set of linear combinations

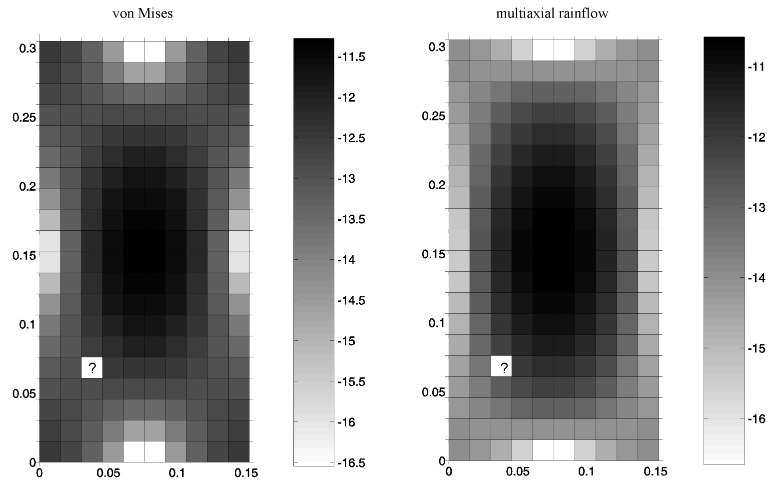

Figure 2: Map of the fatigue damage per time unit of the aluminium rectangular plate

$$s_{mrf}(t) = c_1 s_x(t) + c_2 s_y(t) + c_3 s_{xy}(t) \quad (14)$$

can be chosen for values of c_i such as

$$\sqrt{c_1^2 + c_2^2 + c_3^2} = 1 \quad (15)$$

thus defining a sphere (K.Dressler et al., 1995; A.Beste et al., 1992).

A frequency domain formulation

After defining the two vectors $c = (c_1, c_2, c_3)^T$ and $s = (s_x, s_y, s_{xy})^T$, Equ. (14) can be rewritten

$$s_{mrf} = c^T s \quad (16)$$

The auto-correlation function of $s_{mrf}(t)$ can be computed as follows:

$$E[s_{mrf}(t+\tau)s_{mrf}(t)] = c^T E[s(t+\tau)s^T(t)]c \quad (17)$$

This equation is equivalent to :

$$\begin{aligned} R_{ss_{mrf}}(\tau) &= E[s(t)^T cc^T s(t+\tau)] \\ &= \mathrm{Trace}\{Q^* E[s(t+\tau)s(t)^T]\} \quad (18) \end{aligned}$$

with

$$Q^* = cc^T = \begin{pmatrix} c_1^2 & c_1c_2 & c_1c_3 \\ c_1c_2 & c_2^2 & c_2c_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{pmatrix} \quad (19)$$

Taking the Fourier transform of the autocorrelation function of $R_{ss_{mrf}}(\tau)$, we get the PSD $\Phi_{mrf}(\omega)$ of the scalar process s_{mrf}:

$$\begin{aligned} \Phi_{mrf}(\omega) &= \mathrm{Trace}\{Q^* \Phi_{ss}(\omega)\} \\ &= \sum_{i,j} Q^*_{ij} \Phi_{s_i s_j}(\omega) \quad (20) \end{aligned}$$

where $\Phi_{ss}(\omega)$ is the PSD matrix of the stress vector. Such a scalar process can be defined for every vector c.

The damage is computed for each scalar process Φ_{mrf}, and the most critical combination is determined. Equation (20) shows that the formulation of the equivalent von Mises stress method can be easily generalized to include the multiaxial rainflow method. Both formulations are formally similar, with different definitions of the Q matrix.

5. APPLICATION

The use of both methods is illustrated with a simply supported rectangular aluminium plate ($15.24cm \times 30.48cm$, $e = 0.8mm$) subjected to a band limited white noise random pressure field with perfect spatial coherence [$\Phi_{pp}(\omega) = 1\ Pa^2 sec/rad$, $\omega_c = 6280\, rad/sec$]. The material constants are $c = 4.57\ 10^{55}$, $\beta = 6.09$. The first three modes ($\omega_1 = 663\, rad/sec$, $\omega_2 = 1061\, rad/sec$ and $\omega_3 = 1723\, rad/sec$) are within the bandwidth of the excitation, although the second mode is not excited (because it is anti-symmetrical). 200 shell elements have been used in the discretization.

Results : Figure 2 shows the maps of relative damage per unit time for the Single Moment method respectively established according to the equivalent von Mises stress method and to the multiaxial rainflow method. The gray scale refers to the logarithm of the damage per unit time. Although the life predicted by both methods for a specific element are different, we note that the localization of the damage levels is similar. Figure 4 shows the PSD of the component stresses and that of the equivalent von Mises stress and of the multiaxial rainflow for the element marked with a question mark in Fig.2. The similarity of the two plots is striking. Note that $\Phi_c(\omega)$ as well as $\Phi_{mrf}(\omega)$ always envelope the PSD's of the stress components.

For the selected element, Fig.3 shows the sphere with a gray scale referring to the damage corresponding to each linear combination of the stress components as defined Equ.(14) and Equ.(15). This plot shows how large the difference between maximum and minimum damage is and how the damage depends on the projection (i.e. the elements of the Q^* matrix).

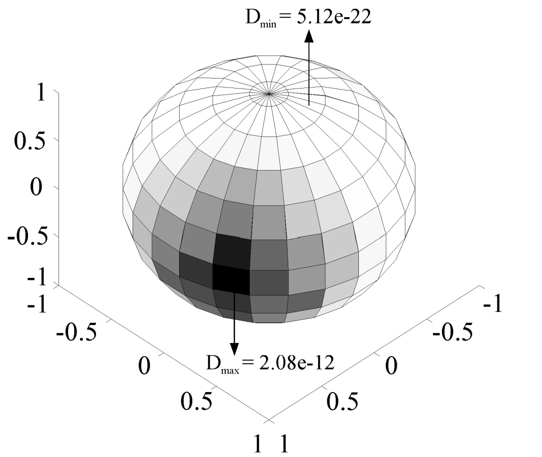

Figure 3: Sphere damage for the selected element

6. STRESS TENSOR HISTORY

For critical elements identified with a frequency domain analysis, the structural analyst may wish to perform more detailed analyses for which he needs representative time histories of the stress vector s_i. Stress-based multiaxial random fatigue life prediction methods have been developed and are now available for industrial applications (e.g. (B.Weber et al., 1997)). The stress tensor histories must have the proper frequency content [described by the diagonal components of $\Phi_{ss}(\omega)$] and the proper cross-correlations [described by the off-diagonal components of $\Phi_{ss}(\omega)$]. The FFT generation of time histories of a given spectral content is a classical problem which is described in textbooks (A.Preumont, 1994). In this section, we discuss its extension to a vector

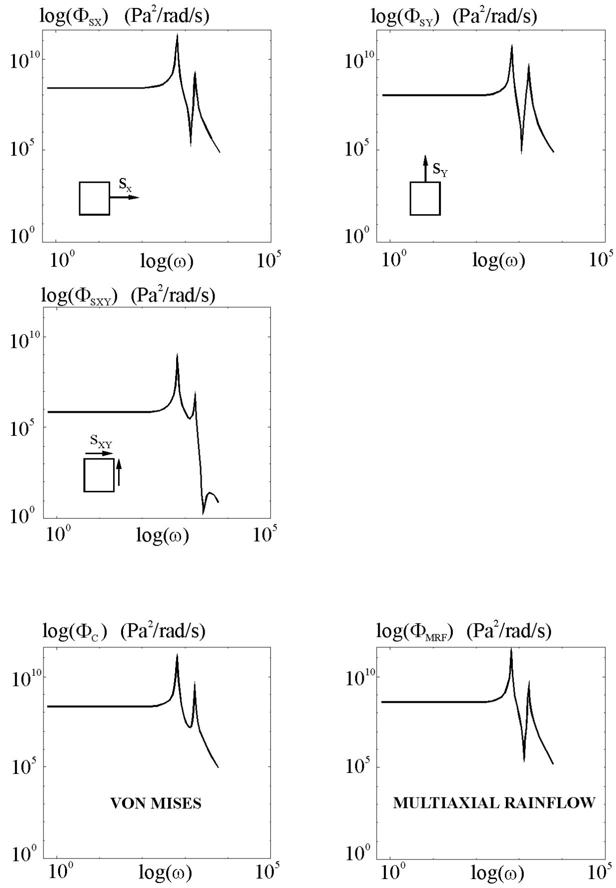

Figure 4: Power spectral densities of the stress components, the von Mises stress and the multiaxial rainflow

process of given PSD matrix $\Phi_{ss}(\omega)$. Recall that $\Phi_{ss}(\omega)$ is Hermitian and semi positive definite. For each frequency, $\Phi_{ss}(\omega)$ can be diagonalized according to

$$\Phi_{ss}(\omega) = U(\omega)\Lambda(\omega)U^H(\omega) \tag{21}$$

where $\Lambda(\omega)$ contains the eigenvalues of $\Phi_{ss}(\omega)$ and $U(\omega)$ its eigenvectors. $\Lambda(\omega)$ is real and diagonal, while the orthogonality of the eigenvectors implies that $U(\omega)$ is unitary ($U^H U = I$).

The diagonal components of $\Lambda(\omega)$ characterize fully independent stochastic processes. The rotation matrix $U(\omega)$ defines the reference frame where the independence is effectively achieved at the frequency ω.

The diagonal matrix $\Lambda(\omega)$ can be transformed into a vector process with independent components. If $Z(\omega, T)$ denotes the Fourier transform of this random vector of finite duration T, it satisfies the relationship:

$$\Lambda(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} \mathrm{E}[Z(\omega,T), Z^H(\omega,T)]. \qquad (22)$$

Combining with Equ.(21) we get:

$$\Phi_{ss}(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T} \mathrm{E}[U(\omega)Z(\omega,T), Z^H(\omega,T_0)U^H(\omega)] \qquad (23)$$

This relationship indicates that if $Z(\omega,T)$ is constructed to be the Fourier transform of the vector process with independent components of PSD matrix $\Lambda(\omega)$,

$$X(\omega,T) = U(\omega)Z(\omega,T) \qquad (24)$$

is the Fourier transform of a vector process of PSD matrix $\Phi_{ss}(\omega)$. The FFT generation of time histories can thus be applied for each vector component.

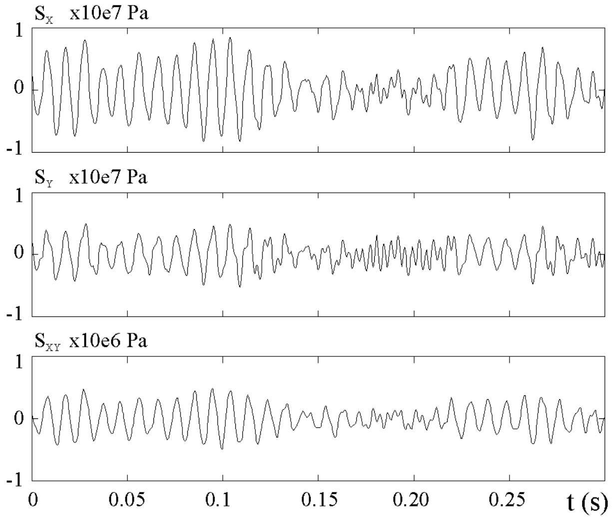

Figure 5: Tensor time history of the selected element

7. CONCLUSION

This paper discusses two methods for direct finite element estimation of fatigue damage from spectral data. The first method is based on the definition of the von Mises stress as a uniaxial random process for which the PSD function is computed. The second one is a frequency domain formulation of the multiaxial rainflow method. Both methods lead to a similar implementation, although the multiaxial rainflow requires the scanning of a large number of linear combination of the stress components. The two approaches give similar damage maps, allowing the designer to localize the most critical elements. The paper also describes a method for the generation of multiaxial stress tensor histories of given PSD matrix, in order to perform a more accurate analysis using methods in the time domain.

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